6th ISAV2016 International Conference on Acoustics and Vibration
7-8 Dec 2016 Tehran - Iran
Iranian Society of Acoustics and Vibration
K. N. Toosi University of Technology

Natural Frequency of a Vertical Pumps Submerged in Fluid

Abbas Dorahaki[a], Ali Najafi Ardakani[b*], Hossein Farzandi[a]

[a] *Master of Science in Mechanical Engineering, South Pars Gas Complex, Assaluyeh, Iran.*

[b] *Faculty of Mechanical Engineering, K.N. Toosi University of Technology, Tehran, Iran.*

** Corresponding author e-mail: najafi@kntu.ac.ir*

Abstract

Vertical pump and motor structures are particularly susceptible to a resonance condition created by the coincidence of the operating RPM[1] with the natural frequency of the pump structure. In this study, vertical pump is modelled by clamped-free cylindrical shell which is submerged in fluid and the effects of the geometry characteristics on natural frequency are investigated. Love's linear strain theory equations with beam functions as axial modal function have been applied to model the shell and boundary condition, respectively. The cantilevered cylindrical shell with open ends is partially submerged in a liquid which is assumed to be incompressible and inviscid, the liquid motion can be described as the velocity potentials written in terms of the appropriate Bessel functions for both the inner and the outer liquid regions. The kinetic energy of the fluid is derived by solving a boundary-value problem related to the fluid motion and potential energy of the shell is derived by plane stress. Finally the natural frequency is obtained using energy functional by the Lagrangian function with Rayleigh-Ritz method. The validity of the theoretical method is compared with other researches. The effects of the submerged depth, fluid type, and geometry characteristics on natural frequency are demonstrated.

Keywords: natural frequency; vertical pump; cylindrical shell; fluid.

---

[1] Rotation per minute

1. Introduction

Centrifugal pumps are among the more versatile and widely used pieces of rotating mechanical equipment found today. Pumps are essential in almost all utilities services and power generation plants. It is estimated that pumps consume approximately 31% of rotating equipment electrical power used throughout industry. Pumps are a vital part of our lives on the planet today.

Centrifugal pumps are subjected to operational forces generated by their operating speed, system head, pressure and piping arrangement. These operational forces cause forced vibration and may originate from the rotating parts or operating conditions. Pump structure and fluid- structure interaction cause free vibration. This vibration reduces the expected life of the pump bearings and other components.

Vibration analysis of vertical pump by condition monitoring devices has been investigated by a large number of researchers but modelling it which is submerged in fluid by theoretical approach and effect of geometry characteristics hasn't been investigated.

Smith studied on the vibration analysis of vertical pump experientially [1]. Beekman investigated resonant vibrations in vertical pumps and showed how it can be managed [2]. Meher studied optimal foundation design of a vertical pump assembly by using finite element method [3]. Baoyun investigated guide bearing probability load theory of large vertical pump and established the probability load theory of the guide bearing for first time [4]. Abdel-Rahman studied on the diagnosis vibration problems of pumping stations [5]. Nikumbe studied on the modal analysis of vertical turbine pump by using finite element method [6].

In the fluid-structure interaction subjects, Amabili obtained free vibration problem of circular cylindrical shells half-filled with liquid analytically [7]. Ergin examined the dynamic characteristics (i.e., natural frequencies and mode shapes) of a partially filled and/or submerged, horizontal cylindrical shell [8]. Sharma developed an analytical investigation of the natural frequency response of multi-layered orthotropic circular cylindrical shells partially submerged in or filled with non-viscous incompressible fluid [9]. Askari studied the dynamic characteristics of a circular cylindrical shell in contact with a liquid theoretically. The cantilevered cylindrical shell with open ends is partially submerged in a liquid [10]. Kwak investigated the free flexural vibration of a hung clamped-free cylindrical shell partially submerged in a fluid [11]. Bae investigated the effects of the boundary conditions such as the existence of the external wall, internal shaft, and bottom on the natural vibration characteristics of the partially submerged cylindrical shell both theoretically and experimentally [12].

However, they did not consider the effect of the hanging vertical pump partially submerged in a liquid on natural frequency. In this study, the effect of external fluids coupled with a partially submerged clamped-free vertical pump that internal space filled of fluid is evaluated. In particular, the effects of the external fluid level with geometric characteristics on natural frequency are investigated in detail.

2. Formulation

2.1 Structural Modelling

In vertical pump body Fig.1, the thickness of body is geometrically neglected in comparison with the radius so it is assumed cylindrical shell and can be applied the theoretical formulation of thin-wall structures. Fig.2 shows the coordinate system of a containing fluid cylindrical shell which is submerged in fluid. The geometrical parameters of the cylindrical shell are mid-surface radius (R), thickness of cylindrical shell (h), and length (L), the external surface of wetted region (L_{fo}) and mass density of fluid (ρ_f). The cylindrical shell is assumed to be thin with a uniform thickness. The displacement components in the x, θ and z directions are denoted by u, v and w, respectively. Young's modulus E, the Poisson's ratio ν, and the mass density ρ are the material parameters where the subscript s and f denotes shell material and fluid, respectively.

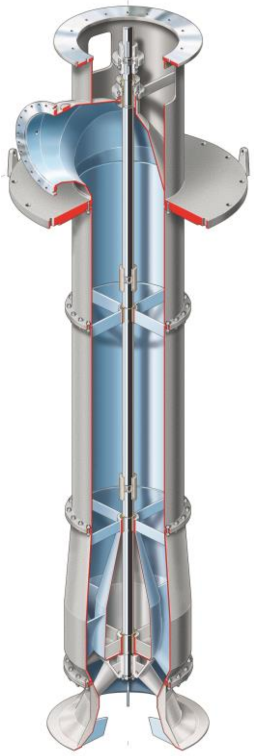

Figure 2. Vertical pump schematic

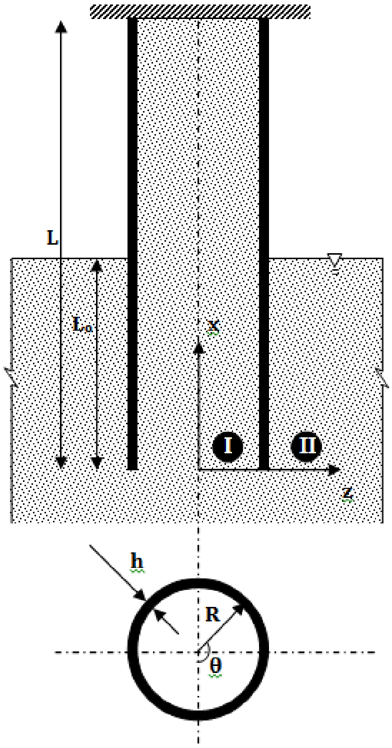

Figure 1. Coordinate system of containing fluid cylindrical shell submerged in fluid

Based on the state of generalized plane stress of shells, the normal stress is assumed to be zero in the radial direction. In this regard, the fundamental equations of the shell for a thin cylindrical shell can be expressed by Hook's law in two dimensional as

$$\{\sigma_S\} = [Q_S]\{\varepsilon\} \tag{1}$$

{σ} and {ε} are the vectors of stress and strain, respectively; and [Q] denote the elastic constants matrix for shell material. The strain vector {ε} in its component form is defined as:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_\theta \\ \varepsilon_{x\theta} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma \end{Bmatrix} + z \begin{Bmatrix} k_1 \\ k_2 \\ \tau \end{Bmatrix} \tag{2}$$

Where ε_1, ε_2 and γ are reference surface strains, and k_1, k_2 and τ are the surface curvatures. The linear deflection and curvatures are defined by Love's theory as [13]:

$$\begin{aligned} &\varepsilon_1 = \frac{\partial u}{\partial x}, \varepsilon_2 = \frac{1}{R}\left(\frac{\partial \upsilon}{\partial \theta} + w\right), \gamma = \frac{1}{R}\left(\frac{\partial u}{\partial \theta}\right) + \frac{\partial \upsilon}{\partial x} \\ &k_1 = -\frac{\partial^2 w}{\partial x^2}, k_2 = \frac{1}{R^2}\left(\frac{\partial v}{\partial \theta} - \frac{\partial^2 w}{\partial \theta^2}\right), \tau = \frac{1}{R}\left(\frac{\partial \upsilon}{\partial x} - \frac{\partial^2 w}{\partial x \partial \theta}\right) \end{aligned} \tag{3}$$

The force and moment resultants for a cylindrical shell are written in the integral forms, respectively.

$$\{N_x, N_\theta, N_{x\theta}\} = \int_{-h/2}^{h/2} \{\sigma_x, \sigma_\theta, \sigma_{x\theta}\}_S \, dz \tag{4}$$

$$\{M_x, M_\theta, M_{x\theta}\} = \int_{-h/2}^{h/2} \{\sigma_x, \sigma_\theta, \sigma_{x\theta}\}_S \, z dz \tag{5}$$

Where N_x, N_θ, and $N_{x\theta}$ are force components in axial, circumferential, and shear directions. M_x, M_θ, and $M_{x\theta}$ are moment components in axial, circumferential, and shear direct-ions. Eqs.(4) and (5) can also be rewritten in the matrix form as:

$$\{N\} = [K]\{\varepsilon\} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}\{\varepsilon\} \tag{6}$$

Where [K] is stiffness matrix and A, B and D are sub-matrices of extensional, coupling, and boundary stiffness, respectively and are given as [13]:

$$[A] = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \quad [B] = \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & B_{66} \end{bmatrix} \quad [D] = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix}$$
$$A_{ij} = \int_{-h/2}^{h/2} Q_{ijs}\,dz \quad B_{ij} = \int_{-h/2}^{h/2} Q_{ijs}\,z\,dz \quad D_{ij} = \int_{-h/2}^{h/2} Q_{ijs}\,z^2\,dz \tag{7}$$

2.2 Governing equations of external and internal fluid

As to apply the effects of internal and external fluid, the kinematic energy must be considered. It is assumed here that the vibrating cylindrical shell is either completely filled with an incompressible in-viscid liquid performing a potential motion. Total potential fluid can be expressed in terms of following:

$$\phi = \phi^i + \phi^e \tag{8}$$

The potential function Φ for a compressible non-viscous liquid must satisfy the following equation [9]:

$$\nabla^2\phi = 0 \tag{9}$$

Eq.(9) reduces to Laplace equation in polar coordinates (r,θ,x), i.e.

$$\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} + \phi_{xx} - \lambda^2{}_m\phi = -\frac{\omega^2}{C^2}\phi \tag{10}$$

Where ω and c are natural frequency and speed of sound respectively. since the shell displacement components have been assumed to be periodic in x and θ it can be assumed that the fluid motion is also periodic with respect to these variables.

$$\phi(r,\theta,x) = \Phi(r)P(x)\cos(n\theta) \tag{11}$$

Where m, n and P(x) are the axial wave numbers, circumferential wave numbers and longitudinal mode shape, respectively. It is assumed that both ends are to be open, so [14]

$$P(x) = \sin(\lambda_m x) \tag{12}$$

The general solution of Eq.(10) in the case of axisymmetric boundary conditions is [14]

$$\Phi(r,\theta) = F(r)\cos(n\theta) \tag{13}$$

By sub-situating for Φ from Eqs.(11), (12) and (13) into the Laplace's equation:

$$\frac{d^2F(r)}{dr^2} + \frac{1}{r}\frac{dF(r)}{dr} - \left[\lambda_m{}^2 - \left(\frac{\omega}{c}\right)^2 + \left(\frac{n}{r}\right)^2\right]F(r) = 0 \tag{14}$$

This clearly is a modified Bessel's equation in Φ with general solution.

For internal fluid given by:

$$\Phi(r) = AI_n\left(\frac{m\pi r}{L}\right) \tag{15}$$

And for external fluid given by:

$$\Phi(r) = AI_n\left(\frac{m\pi r}{L}\right) \tag{16}$$

Here I_n and K_n are the modified Bessel function of order n and of first and second kind, respectively. If both ends of the fluid volume are assumed to be open, so that a zero pressure is assumed there, the boundary conditions

$$\phi\big|_{x=0} = \phi\big|_{x=L} = 0 \tag{17}$$

When the thickness of the shell is assumed to be thin, the radial liquid velocity along the inner wetted shell surface must be identical to the radial velocity of the flexible shell [11] [12], so

For internal fluid boundary condition:

$$\left.\frac{\partial \phi}{\partial r}\right|_{r=R} = \frac{\partial w}{\partial t} \tag{18}$$

And for external fluid boundary condition:

$$\left.\frac{\partial \phi}{\partial r}\right|_{r=R} = -\frac{\partial w}{\partial t} \tag{19}$$

2.3 Lagrange energy equations

Considering the displacements as periodic functions, they can be expressed in Fourier series form as [15]:

$$\begin{aligned} u(x,\theta,t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_{mn} \frac{d\Gamma}{dx} \cos(n\theta) e^{i\omega_{mn} t} \\ \upsilon(x,\theta,t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V_{mn} \Gamma(x) \sin(n\theta) e^{i\omega_{mn} t} \\ w(x,\theta,t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn} \Gamma(x) \cos(n\theta) e^{i\omega_{mn} t} \end{aligned} \tag{20}$$

Where U_{mn}, V_{mn} and W_{mn} are constants, and ω_{mn} is natural frequency. The axial modal function $\Gamma(x)$ in Eq.(20) is chosen to satisfy condition at both ends of the cylindrical shell. The beam modal function has been chosen as the axial modal function and expressed [16].

$$\Gamma(x) = \alpha_1 \cosh\left(\frac{\lambda_m x}{L}\right) + \alpha_2 \cos\left(\frac{\lambda_m x}{L}\right) - \zeta_m \left(\alpha_3 \sinh\left(\frac{\lambda_m x}{L}\right) + \alpha_4 \sin\left(\frac{\lambda_m x}{L}\right)\right) \tag{21}$$

For clamped-free condition, there is the values factor in table 1.

Table 1. Values of the axial modal function constants for C-F boundary condition

α_i	λ_m	ζ_m
$\alpha_1=\alpha_3=1$ $\alpha_2=\alpha_4=-1$	$\cos\lambda_m \cosh\lambda_m = -1$	**$(\sinh\lambda_m - \sin\lambda_m)/(\cosh\lambda_m + \cos\lambda_m)$**

The energy method developed by Ritz applies the principle of minimum potential energy and minimum kinetic energy. To determine the natural frequency of vibration for cylindrical shell, the Rayleigh-Ritz method is used. The energy functional (Π) defined by the Lagrangian function for vibration of cylindrical shell.

$$\Pi = T_{max} - U_{max} \tag{22}$$

The total kinetic energies of the system can be represented as the sum of the kinetic energies of the shell, internal fluid and external fluid.

$$T = T_s + T_{\phi^i} + T_{\phi^e} \tag{23}$$

The kinetic energy of the shell is given by [15]:

$$T_s = \int_0^L \int_0^{2\pi} \int_{-h/2}^{h/2} \rho_s \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial \upsilon}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2 \right] R\, dz\, d\theta\, dx \tag{24}$$

The kinetic energy of the internal and external fluid is given by [11] [12]:

$$T_{\phi^i} = \frac{1}{2} \rho_f R \int_0^L \int_0^{2\pi} -\phi^i \dot{w}\, d\theta\, dx \tag{25}$$

$$T_{\phi^e} = \frac{1}{2} \rho_f R \int_{L-L_o}^{L} \int_0^{2\pi} \phi^e \dot{w}\, d\theta\, dx \tag{26}$$

Where L_o is external fluid level. The potential energy of shell is given by [15]:

$$U = \frac{1}{2}\int_0^L \int_0^{2\pi} \{N_s\}^T \{\varepsilon\} R d\theta dx \tag{27}$$

3. Results and discussion

By substituting Eqs.(3) and (6) into (24) and also Eqs.(11) and (16) into Eqs.(24), (25) and (26) and applying Rayleigh-Ritz method with minimizing the energy functional Π with respect to the unknown coefficients as follows:

$$\frac{\partial \Pi}{\partial U} = \frac{\partial \Pi}{\partial V} = \frac{\partial \Pi}{\partial W} = 0 \tag{28}$$

Thus, a set of equation that includes three equations of motion for cylindrical shells submerged in fluid is obtained. The three governing eigen-values of the equations of motion can be expressed in matrix form as:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix} = \omega^2 \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix} \tag{29}$$

In this paper, these equations are solved using MATLAB coding software and the smallest of the three positive roots is the natural frequency characteristic of interest in the present study. These benchmark shells have the characteristics in tables 2.

Table 2. Cylindrical shell characteristics

Length	Radius	Thickness
6.6 m	0.4 m	**0.01 m**
Density	**Elasticity Module**	**Poisson ratio**
8166 kg/m3	2.08×10^{11}	**0.318**

Numerical results on table 3 for a clamped-free aluminium circular cylindrical shell having the following dimension and material properties: L=0.30531 m, R=0.073914 m, h=0.000178 m, ν=0.3, E=6.8258×10^{10} Pa, and ρ=2712.2 kg/m^3. Very good agreement can be seen.

Table 3. Comparison of the natural frequencies obtained by the present approach with those reported by Kurylov [17]

Result	n=2	n=3	n=4	n=5	n=6	n=7	n=8	n=9
Present study	457.1	234.2	175.2	205.1	279.5	377.9	495.2	628.8
Ref [17]	456.4	233.9	175.7	205.3	279.4	377.6	494.1	627.1

As shown in Fig.3, the variation of the natural frequency for n= [1-10] as circumferential wave number and m= [1-6] as axial wave number which it is filled fluid. It can be readily seen from Fig.3 that the fundamental frequency is found to be [n, m] = [1, 1] (ω_{11}=65.83 Hz). Natural frequency converge as the number of admissible functions increases. Based on the numerical results, five admissible functions can be chosen for the numerical analysis.

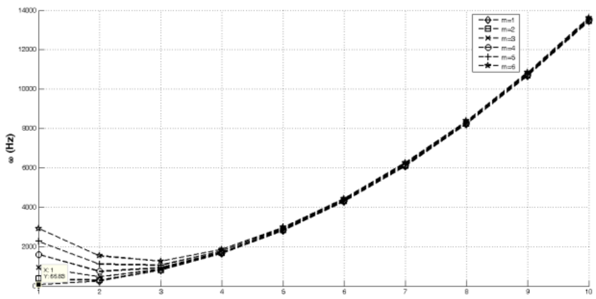

Figure 3. Variations of natural frequencies C-F fluid-filled cylindrical shell without external fluid

In following, the effects of geometry characteristics and external fluid level on the natural frequency are shown. With according to vertical pumps operation, it is assumed that inside the fluid-filled shell and external fluid level is constant in suction status for all calculation.

As shown in Fig.4, the natural frequencies for the various external fluid levels decrease for fixed circumferential wave number but they increase with increasing the circumferential wave number n for the fixed axial mode number m due to the *separation effect*, as depicted in Refs. [10] and [18]. The *separation effect* means that an increase of the circumferential modes produces the nodal points along the periphery of the shell and it divides the circumferential liquid flow near the shell. At the same time, it reduces the highly vibrated surface of the shell. As a result, the effect of the liquid inertia decreases.

The natural frequencies of the cylindrical shell as a function of the submerged depth ratio (L_{fo}) for m= [1, 3] as axial mode number as shown in Figs.5-7. It can be observed in the figures that the wet natural frequencies decrease with the submerged depth ratio. Therefore, the lowest frequencies occur for the fully submerged case.

As shown in Fig.5, the natural frequencies in first circumferential mode decrease rapidly even for the small fluid heights. If the fluid height is more than half of the shell length, then the natural frequencies vary slowly.

As shown in Figs.6 and 7, the change of the natural frequency is not uniform due to the mode shape and nodal points. It is observed that the number of the *transition plateaus* in the natural frequencies increase, one by one with the axial mode number m. The *transition plateaus* means that as the axial nodal points increase, the fluid particles moving in vertical direction will be separated into up and down directions by the number of axial nodal points of the shell [10][18]. As a result, that the number of transition plateaus of natural frequencies is identical to that of axial nodal points of the shell. It is shown in Fig.8.

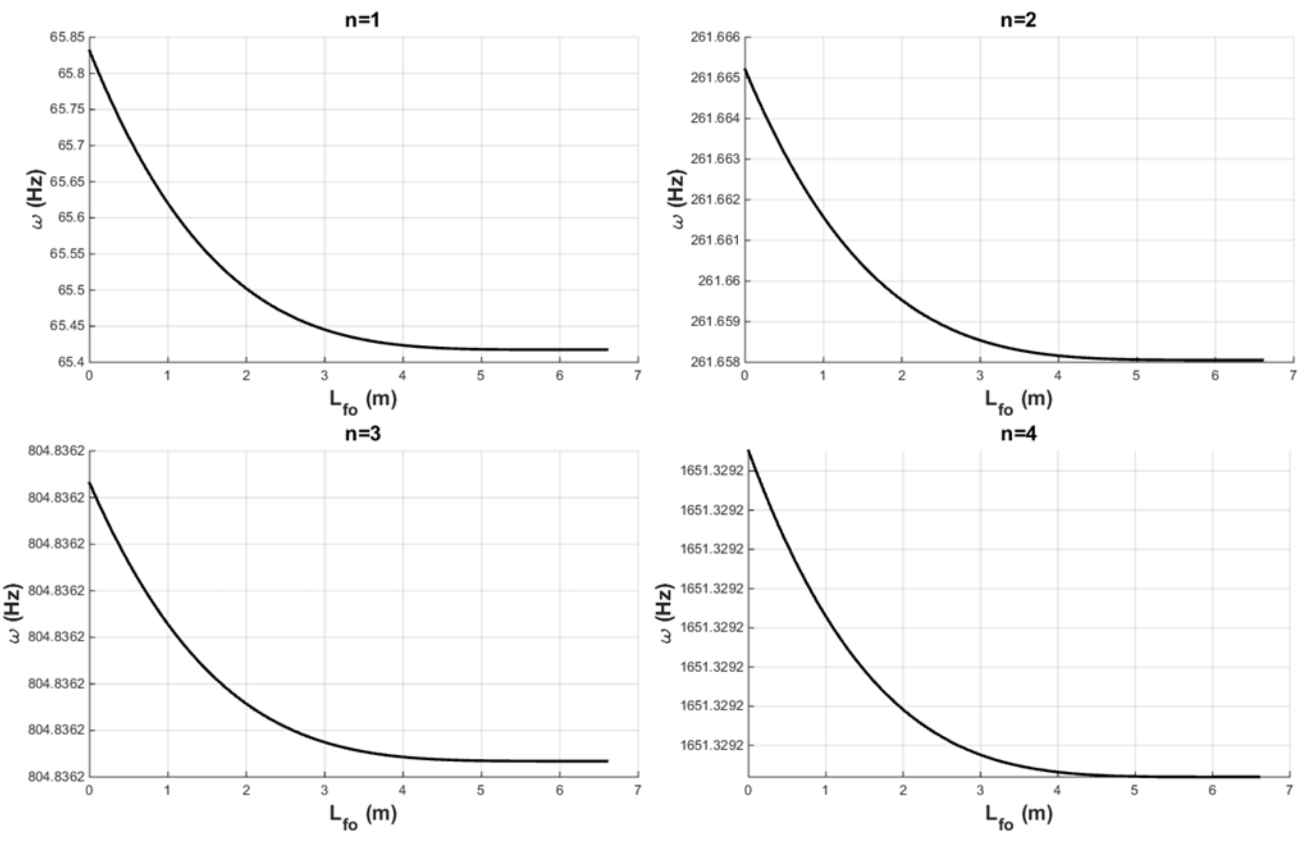

Figure 4. Variations of natural frequencies C-F fluid-filled cylindrical shell with various external fluid level for n= [1, 4]

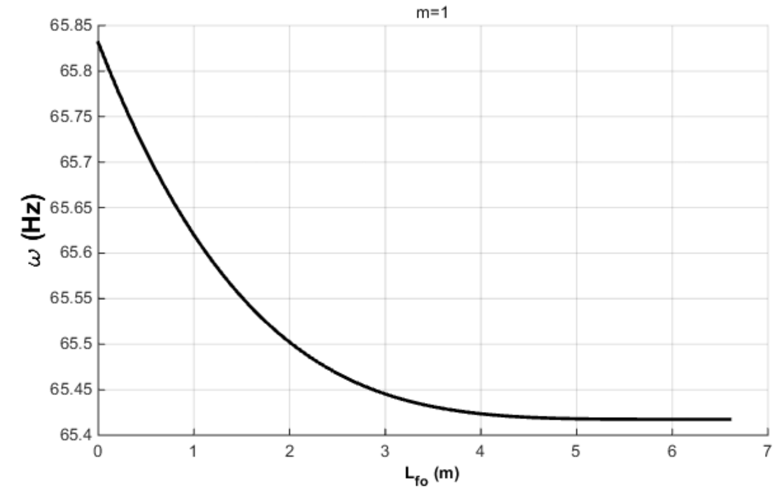

Figure 5. Effect of the submerged depth ratio (L_{fo}) on the fluid-filled natural frequency of the cylindrical shell for the first axial mode m=1

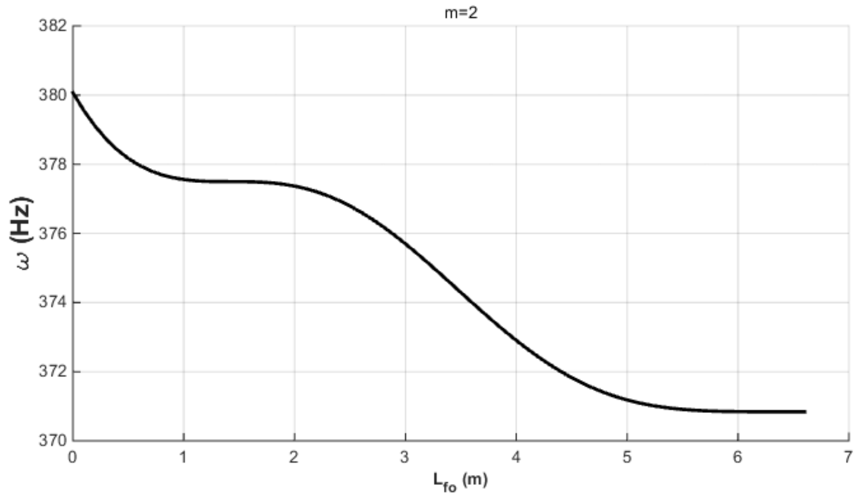

Figure 6. Effect of the submerged depth ratio (L_{fo}) on the fluid-filled natural frequency of the cylindrical shell for the second axial mode m=2

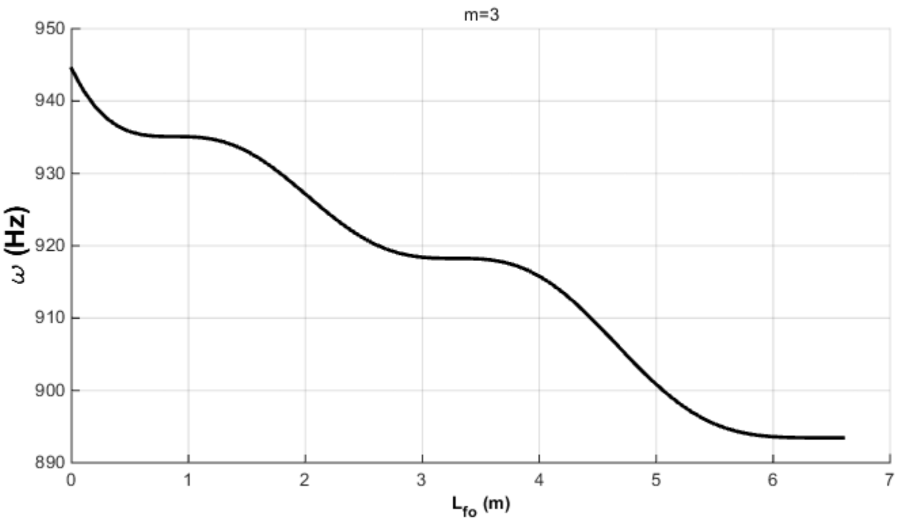

Figure 7. Effect of the submerged depth ratio (L_{fo}) on the fluid-filled natural frequency of the cylindrical shell for the third axial mode m=3

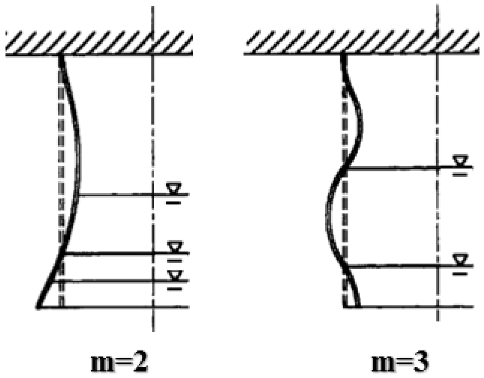

Figure 8. Effect of the axial mode shapes and liquid levels on the transition plateaus phonumena Ref.[18]

As shown in Fig.9, the fundamental natural frequency increase by increasing the radius of shell. The effects of external fluid level on the fundamental frequency are very little for fixed radius and the fundamental natural frequency decrease by increasing external fluid level.

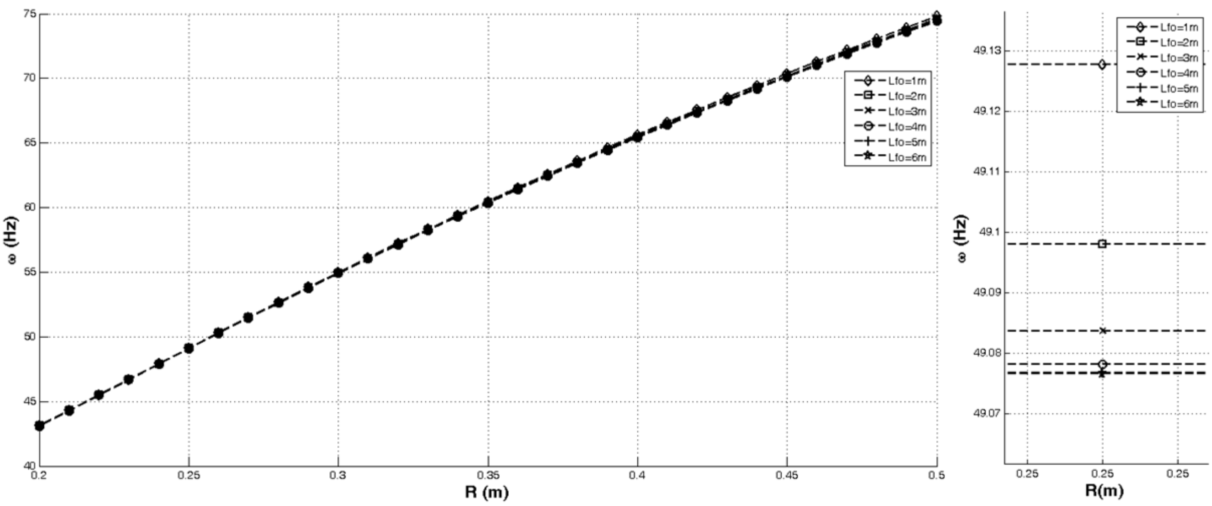

Figure 9. Variations of fundamental natural frequency C-F fluid-filled cylindrical shell with various external fluid level and radius of shell

As shown in Fig.10, the fundamental natural frequency decrease by increasing the length of shell. In fact the kinematic energy of internal fluid increase when the length of shell increase.

As shown in Fig.11, the fundamental natural frequency increase by increasing the thickness of shell since the thickness increase the stiffness of shell.

4. Conclusion

In this paper, the free vibrations of clamped-free fluid-filled vertical pump are analysed. As this study, the vertical pump is modelled by cylindrical shell. Love's linear large deflection theory is applied to model the cylindrical. The shell deformation is described in terms of longitudinal, circumferential and radial displacement fields. A semi analytical approach, where in the displacement fields are expanded by means of a double mixed series based on harmonic functions for the longitudinal, circumferential and radial variables, is proposed to characterize the linear response of the cylindrical shell.

The properties of fluid are assumed to be in-viscid, ir-rotational, and incompressible with external fluid level L_o and mass density ρ_f. Velocity potential of the fluid and finally potential and kinetic energy are obtained. The natural frequencies of cylindrical shells are obtained for linear analysis and compared with the available results in the literature and a good agreement seen.

The numerical example that amounts to a real vertical pump shows that the natural frequencies increase with increasing the circumferential wave number n for the fixed axial mode number m due to the separation effect. The wet natural frequencies decrease with the submerged depth ratio. Therefore, the lowest frequencies occur for the fully submerged case. It is observed that the number of the transition plateaus in the natural frequencies increase, one by one with the axial mode number m. Also the effect of geometry characteristics shell on the natural frequency is found.

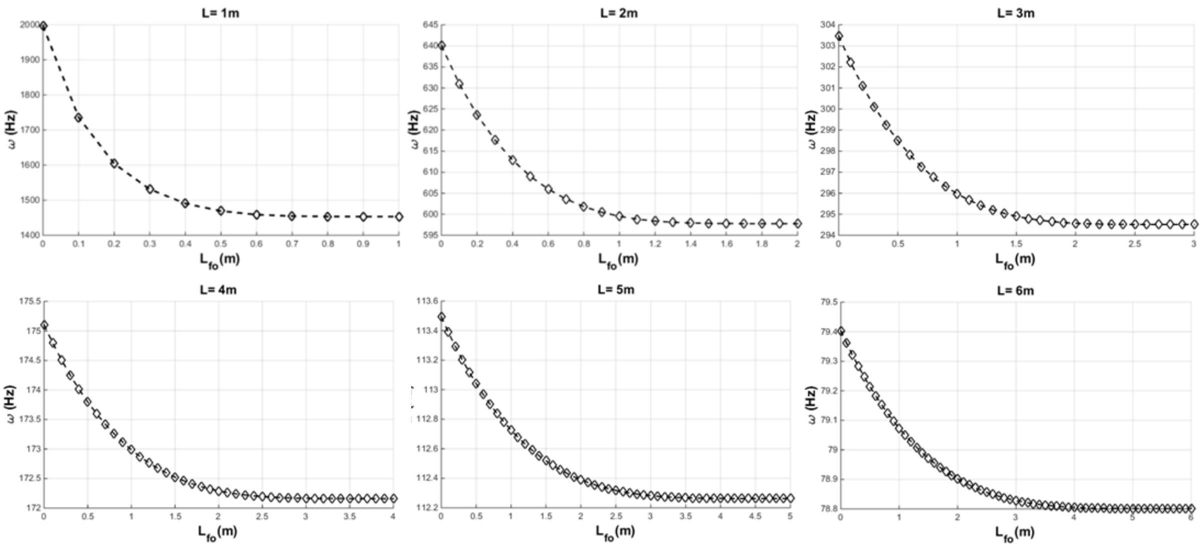

Figure 10. Variations of fundamental natural frequency C-F fluid-filled cylindrical shell with various external fluid level and length of shell

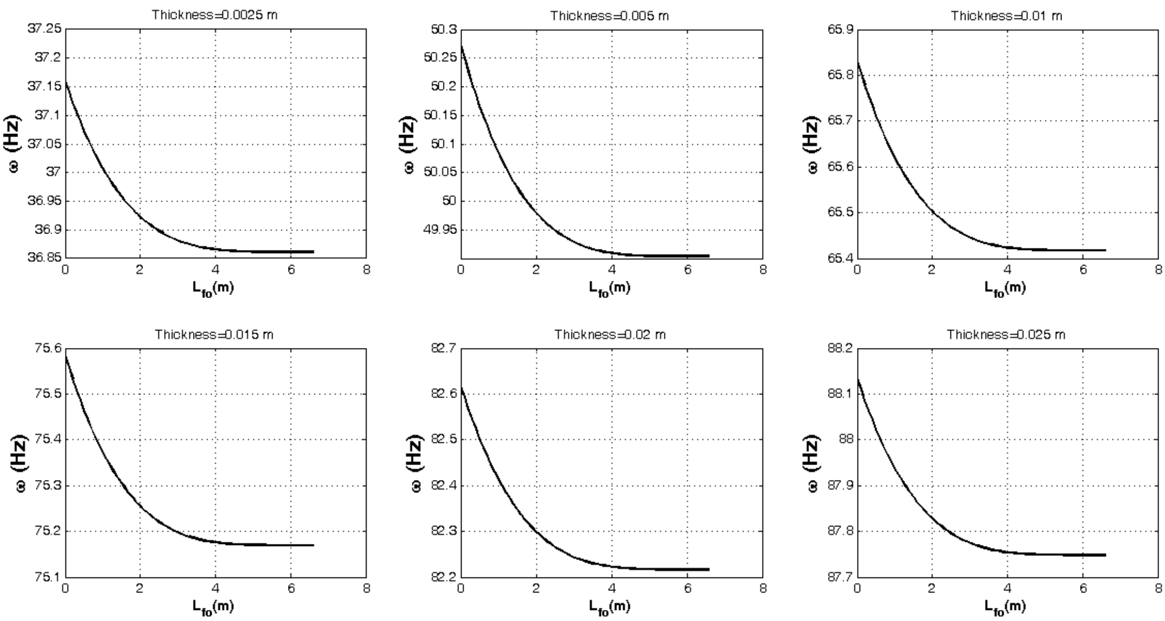

Figure 11. Variations of fundamental natural frequency C-F fluid-filled cylindrical shell with various external fluid level and thickness of shell

REFERENCES

1. D. R. Smith, G. M. Woodward, "Vibration analysis of vertical pump", *sound and vibration*, 1988, pp. 24-30.
2. B. Beekman, "Resonant vibrations in vertical pumps", World Pumps, 2014, pp. 18-22.
3. Kishor K. Meher, A. Rama Rao," Optimal foundation design of a vertical pump assembly", *Journal of Sound and Vibration*, vol. 291, 2006,pp. 1269–1277.
4. Qiu Baoyun, Lin Haijiang, Yuan Shouqi," Guide bearing probability load theory of large vertical pump", *Mechanism and Machine Theory*, Vol42, 2007, pp. 1199–1209.
5. S. M. Abdel-Rahman, Sami A. A. El-Shaikh,"DIAGNOSIS VIBRATION PROBLEMS OF PUMPING STATIONS", *Thirteenth International Water Technology Conference*, IWTC 13, 2009, pp.419-434.
6. A.Y.Nikumbe, V.G.Tamboli, H.S.Wagh," Modal Analysis of Vertical Turbine Pump", *International Advanced Research Journal in Science, Engineering and Technology* Vol. 2, 2015, pp.217-221.
7. M. AMABILI," FREE VIBRATION OF PARTIALLY FILLED HORIZONTAL CYLINDRICAL SHELLS", *Journal of Sound and Vibration*, vol. 191. 1996. pp.646-679.
8. A. ERGIN, P. TEMAREL, " FREE VIBRATION OF A PARTIALLY LIQUID-FILLED AND SUBMERGED, HORIZONTAL CYLINDRICAL SHELL", *Journal of Sound and Vibration*, vol. 254. 2002. pp.951-965.
9. C. B. Sharma."Natural frequency response of vertical cantilever composite shells containing fluid". *Engineering Structures*. Vol. 20. 1995. pp. 732-737.
10. Ehsan Askari, Kyeong-Hoon Jeong,"Hydroelastic vibration of a cantilever cylindrical shell partially submerged in a liquid", *Ocean Engineering*. vol. 37. 2010. pp.1027–1035.
11. Moon K. Kwak, Jae-Ryang Koo, Chun-Hee Bae, "Free vibration analysis of a hung clamped-free cylindrical shell partially submerged in fluid", *Journal of Fluids and Structures*. vol. 27 .2011. pp. 283–296.
12. Chun-Hee Bae, Moon K. Kwak, Jae-Ryang Koo, "Free vibration analysis of a hanged clamped-free cylindrical shell partially submerged in fluid: The effect of external wall, internal shaft, and flat bottom", *Journal of Sound and Vibration*. vol. 331. 2012. pp.4072–4092.
13. M. R. Isvandzibaei, Hishamuddin Jamaluddin, Raja Ishak Raja Hamzah, "Natral frequency characteristics of thin-walled homogeneous and manifold layered cylindric al shells under pressure using energy method". *J. Cent. South Univ.* vol. 21. 2014. pp. 521−532.
14. Marco Amabili, "Nonlinear Vibrations and Stability of Shells and Plates*", Cambridge University Press.* 2008. pp.150-163
15. A.A. Jafari, S.M.R. Khalili, M.Tavakolian, " Nonlinear vibration of functionally graded cylindrical shells embedded with apiezoelectric layer". *Thin-Walled Structures*. vol. 79. 2014. pp.8–15.
16. M.N.Naeem, A.G.Khan1, S.H.Arshad, A.G. Shah, M.Gamkhar,"Vibration of Three-Layered FGM Cylindrical Shells with Middle Layer of Isotropic Material for Various Boundary Conditions". *World Journal of Mechanics*. vol. 4. 2014. pp.315-331..
17. Ye. Kurylov, M. Amabili, "Nonlinear vibrations of clamped-free circular cylindrical shells", *Journal of Sound and Vibration*, vol.330, 2011, pp.5363–5381.
18. K. H. Jeongt, S. C. Lee, " HYDROELASTIC VIBRATION OF A LIQUID-FILLED CIRCULAR CYLINDRICAL SHELL", computers & structures, vol.66, 1998, pp.173-185.